WS 8.7,9.1

1 WS Problem 1

1. Explain why, and where, the following integrals are improper.
(a) $\int_0^{\pi} \frac{\sin x}{\sqrt{1+\cos x}}\, dx$.

(b) $\int_0^{\pi/2} \sec^2 \theta d\theta$

(c) $\int_1^{\infty} \frac{1}{x(x+1)}\, dx$

(d) $\int_1^{\infty} \frac{1}{t^2-2t+1}\, dt$

(a) $\int_0^{\pi} \frac{\sin x}{\sqrt{1+\cos x}}\, dx$. This integral is improper when $\sqrt{1+\cos x} = 0$, and this happens at $x = \pi$ in $[0, \pi]$ (that's when $\cos x = -1$).

(b) $\int_0^{\pi/2} \sec^2 \theta d\theta$. This integral is improper when $\cos\theta = 0$ in $[0, \pi/2]$, since $\sec^2\theta = 1/\cos^2\theta$. This only happens at $\theta = \pi/2$.

(c) $\int_1^{\infty} \frac{1}{x(x+1)}\, dx$. This integral is improper at the upper bound (∞). Note that the divisions by zero occur at $x = 0, x = -1$, outside our interval $[1, \infty)$, so this integral is only improper at ∞ as usual.

(d) $\int_1^{\infty} \frac{1}{t^2-2t+1}\, dt$. This integral is improper at both bounds. It's improper at $t = 1$ since $t^2 - 2t + 1 = (t-1)^2$, so the integrand $1/(t^2 - 2t + 1)$ has a division by zero at $t = 1$. It's also improper at the upper bound since it is infinite.

2 WS Problem 2

2. Let $f(x) = \sin x$. We want to find $p_8(x)$.
(a) Find $f(x)$ and $f(0)$, $f(x)$ and $f(0)$, ..., $f^{(8)}(x)$ and $f^{(8)}(0)$.

(b) Using (a) and the formula for $p_n(x) = f(0)+f'(0)x+f''(0)/2x^2+...+\frac{f^{(n)}(0)}{n!}x^n$, write down $p_0(x), p_1(x), ..., p_8(x)$.

(c) What pattern do you see in your answer to (b)?

(d) Using the result of (c), write down the term with highest degree of x in each of the following:

(i) $p_{34}(x)$

(ii) $p_{51}(x)$

(iii) $p_{99}(x)$

(a) Find $f(x)$ and $f(0)$, $f(x)$ and $f(0)$, ..., $f^{(8)}(x)$ and $f^{(8)}(0)$.

We have $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = \sin x, f^{(5)}(x) = \cos x, f^{(6)}(x) = -\sin x, f^{(7)}(x) =$

$-\cos x$, $f^{(8)}(x) = \sin x$, so $f(0) = 0 = f''(0) = f^{(4)}(0) = f^{(6)}(x) = f^{(8)}(0) = \pm \sin 0$.

We also have $f'(0) = 1 = f^{(5)}(0) = \cos 0$ and $f'''(0) = -1 = f^{(5)}(0) = -\cos 0$.

(b) Using (a) and the formula for $p_n(x) = f(0) + f'(0)x + f''(0)/2x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n$, write down $p_0(x), p_1(x), ..., p_8(x)$.

We have

$$p_0(x) = f(0) = 0 \quad (1)$$
$$p_1(x) = f(0) + f'(0)x = x \quad (2)$$
$$p_2(x) = p_1(x) + \frac{f^{(2)}(0)}{2!}x^2 = x \quad (3)$$
$$p_3(x) = p_2(x) + \frac{f^{(3)}(0)}{3!}x^3 = x - x^3/3! = x - x^3/6 \quad (4)$$
$$p_4(x) = p_3(x) + \frac{f^{(4)}(0)}{4!}x^4 = x - x^3/6 \quad (5)$$
$$p_5(x) = p_4(x) + \frac{f^{(5)}(0)}{5!}x^5 = x - x^3/6 + x^5/5! = x - x^3/6 + x^5/120 \quad (6)$$
$$p_6(x) = p_5(x) + \frac{f^{(6)}(0)}{6!}x^6 = x - x^3/6 + x^5/120 \quad (7)$$
$$p_7(x) = p_6(x) + \frac{f^{(7)}(0)}{7!}x^7 = x - x^3/6 + x^5/120 - x^7/7! = x - x^3/6 + x^5/120 - x^7/5040 \quad (8)$$
$$p_8(x) = p_7(x) + \frac{f^{(8)}(0)}{8!}x^8 = x - x^3/6 + x^5/120 - x^7/7! = x - x^3/6 + x^5/120 - x^7/5040 \quad (9)$$
$$(10)$$

(c) What pattern do you see in your answer to (b)?

We have odd powers with alternating sign, and for even numbers n we have $p_n = p_{n-1}$: formally,

$p_{2n+1}(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$, for $n \geq 0$, and $p(0) = 0$, with $p_{2n} = p_{2n-1}$ for $n \geq 1$.

(d) Using the result of (c), write down the term with highest degree of x in each of the following:

(i) $p_{34}(x)$

With our pattern, we notice $p_{34} = p_{34-1} = p_{33} = p_{2(16)+1} = \sum_{i=0}^{16} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$. The highest term in x occurs when $i = 16$:

$$\frac{(-1)^{16} x^{2(16)+1}}{(2(16)+1)!} = x^{33}/(33!)$$

(ii) $p_{51}(x)$

Using our pattern, $p_{51} = p_{2(25)+1} = \sum_{i=0}^{25} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$. The highest term in x occurs when $i = 25$:

$$\frac{(-1)^{25} x^{2(25)+1}}{(2(25)+1)!} = -x^{51}/(51!)$$

(iii) $p_{99}(x)$

Using our pattern, $p_{99} = p_{2(49)+1} = \sum_{i=0}^{49} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$. The highest term in x occurs when $i = 49$:

$$\frac{(-1)^{49} x^{2(49)+1}}{(2(49)+1)!} = -x^{99}/(99!)$$

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3 Ch. 8 Review

4 Review Problem 1

Evaluate:

(A) $\int t^2 \sqrt{1-3t}\, dt$

We'll use the tabular method (integration by parts) since we are dealing with a polynomial (t^2) multiplied by something we can integrate ($\sqrt{1-3t}$):

u	dv
t^2	$\sqrt{1-3t}$
$2t$	$-\frac{2}{9}(1-3t)^{3/2}$
2	$\frac{4}{135}(1-3t)^{5/2}$
0	$-\frac{8}{2835}(1-3t)^{7/2}$

The u column is derivatives of t^2.

To find the dv column: below $\sqrt{1-3t}$ we have $v = \int \sqrt{1-3t}\, dt = \frac{-1}{3}\int \sqrt{u}\, du = [\frac{-1}{3} \cdot \frac{2}{3}(1-3t)^{3/2}] = [\frac{-2}{9}(1-3t)^{3/2}]$ for $u = 1-3t$, $du = -3\, dt$.

Next, below $\frac{-2}{9}(1-3t)^{3/2}$ we have $\int \frac{-2}{9}(1-3t)^{3/2}\, dt = \frac{-2}{9}\int(1-3t)^{3/2}\, dt = \frac{-2}{9}(\frac{-1}{3}\int u^{3/2}\, du) = \frac{2}{27} \cdot \frac{2}{5}[u^{5/2}] = [\frac{4}{135}(1-3t)^{5/2}]$.

Next, below $\frac{4}{135}(1-3t)^{5/2}$ we have $\int \frac{4}{135}(1-3t)^{5/2} = \frac{4}{135}(\frac{-1}{3}\int u^{5/2}\, du) = \frac{-4}{405}[\frac{2}{7}u^{7/2}] = [\frac{-8}{2835}(1-3t)^{7/2}]$.

Now add row 1 of the u column times row 2 of the dv column and so on, alternating signs and adding them all together:

$$\int t^2\sqrt{1-3t}\, dt = t^2(\frac{-2}{9}(1-3t)^{3/2}) - 2t(\frac{4}{135}(1-3t)^{5/2}) + 2(\frac{-8}{2835}(1-3t)^{7/2}) + C \quad (11)$$

$$= \frac{-2t^2}{9}(1-3t)^{3/2} - \frac{8t}{135}(1-3t)^{5/2} - \frac{16}{2835}(1-3t)^{7/2} + C \quad (12)$$

(B) $\int \frac{t^7}{(1-t^4)^3}\, dt$

For this integral, we could solve it quickly using a u-substitution:

if $u = 1 - t^4$, then $du = -4t^3\ dt$, so $\frac{-1}{4} du = t^3\ dt$. Also, $t^4 = 1 - u$:

$$\begin{aligned}
\int \frac{t^7}{(1-t^4)^3}\,dt &= \int \frac{t^4 \cdot t^3}{(1-t^4)^3}\,dt && (13)\\
&= \int \frac{(1-u)\cdot -1/4 \cdot du}{u^3} && (14)\\
&= -1/4 \int \frac{1}{u^3} - \frac{u}{u^3}\,du && (15)\\
&= -1/4[\frac{1}{-2}u^{-2} - \frac{1}{-1}u^{-1}] && (16)\\
&= \frac{1}{8u^2} - \frac{1}{4u} + C && (17)\\
&= \frac{1}{8(1-t^4)^2} - \frac{1}{4(1-t^4)} + C && (18)
\end{aligned}$$

Or we can solve it the long way using partial fractions:

$$\begin{aligned}
\frac{t^7}{(1-t^4)^3} &= \frac{t^7}{(1+t^2)^3(1-t^2)^3} && (19)\\
&= \frac{t^7}{(1+t^2)^3(1-t)^3(1+t)^3} && (20)
\end{aligned}$$

So,

$$\frac{t^7}{(1+t^2)^3(1-t)^3(1+t)^3} = \frac{Ax+B}{1+t^2} + \frac{Cx+D}{(1+t^2)^2} + \frac{Ex+F}{(1+t^2)^3} + \frac{G}{1-t} + \frac{H}{(1-t)^2} + \frac{I}{(1-t)^3} + \frac{J}{1+t} + \frac{K}{(1+t)^2} + \frac{L}{(1+t)^3}$$

We won't solve this here.

$(C) \int \frac{2x+4}{x^2+4x-5}\,dx$

To solve this, we'll do a u-sub for $u = x^2 + 4x - 5$, so $du = 2x + 4\ dx$:
$\int \frac{2x+4}{x^2+4x-5}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2 + 4x - 5| + C$.

5 Review Problem 2

Determine whether the integral is proper or improper. Evaluate the integrals:
$(A) \int_0^{\pi/3} \tan^5 x \sec x\ dx$

This integral is proper: we can rewrite $\tan^5 x \sec x = \frac{\sin^5 x}{\cos^6 x}$, and $\cos x$ is nonzero on $[0, \pi/3]$.

To evaluate the integral, split off a $\tan x$ and set $u = \sec x$, so $du = \sec x \tan x \, dx$:

$$\begin{aligned}
\int_0^{\pi/3} \tan^5 x \sec x \, dx &= \int_0^{\pi/3} \tan^4 x \sec x \tan x \, dx && (21)\\
&= \int_0^{\pi/3} (\sec^4 x - 1) \sec x \tan x \, dx && (22)\\
&= \int_1^2 u^4 - 1 \, du && (23)\\
&= [\frac{1}{5}u^5 - u]_1^2 && (24)\\
&= 2^5/5 - 2 - (1/5 - 1) && (25)\\
&= (32 - 10 - 1 + 5)/5 = 26/5 && (26)
\end{aligned}$$

(B) $\int_0^1 x \ln x \, dx$

This integral is improper at one of our bounds. The value $x = 0$ is outside the domain of the integrand since $\ln 0$ is undefined.

Thus we begin our improper integral. First, note we may evaluate $\int x \ln x \, dx$ with an integration by parts with $u = \ln x$, $dv = x \, dx$, so $du = \frac{1}{x} \, dx$, $v = x^2/2$:

$$\begin{aligned}
\int x \ln x \, dx &= x^2/2 \cdot \ln x - \int x^2/2 \cdot \frac{1}{x} \, dx && (27)\\
&= x^2/2 \cdot \ln x - \int x/2 \, dx && (28)\\
&= x^2 \ln(x)/2 - x^2/4 + C && (29)
\end{aligned}$$

Then, we have

$$\begin{aligned}
\int_0^1 x \ln x \, dx &= \lim_{a \to 0^+} [x^2 \ln(x)/2 - x^2/4]_a^1 && (30)\\
&= \lim_{a \to 0^+} \ln(1)/2 - 1/4 - (a^2 \ln(a)/2 - a^2/4) && (31)\\
&= \lim_{a \to 0^+} -1/4 - a^2 \ln(a)/2 && (32)
\end{aligned}$$

We can evaluate the indeterminate form $0 \cdot -\infty$ of $\lim_{a \to 0^+} a^2 \ln(a)/2$ using l'Hopital's rule:

$$\begin{aligned}
\lim_{a \to 0^+} a^2 \ln(a)/2 &= \lim_{a \to 0^+} \frac{\ln(a)}{2\frac{1}{a^2}} && (33)\\
&= \lim_{a \to 0^+} \frac{1/a}{-4\frac{1}{a^3}} && (34)\\
&= \lim_{a \to 0^+} \frac{a^2}{-4} = 0 && (35)
\end{aligned}$$

Thus $\int_0^1 x \ln x \, dx = \lim_{a \to 0^+} -1/4 - a^2 \ln(a)/2 = -1/4$

$(C)\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\,dx$

Note that the integral is improper only at the endpoint $x=1$ since the denominator $\sqrt{1-x^2}$ is zero there.

We calculate an antiderivative using the trig substitution $x=\sin u$, $dx=\cos u\,du$:

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^3 u\cos u}{\sqrt{1-\sin^2 u}}\,du && (36)\\
&= \int \frac{\sin^3 u\cos u}{\sqrt{\cos^2 u}}\,du && (37)\\
&= \int \sin^3 u\,du && (38)\\
&= \int \sin^2 u\sin u\,du && (39)\\
&= \int (1-\cos^2 u)\sin u\,du && (40)\\
&= \int (1-w^2)\cdot - dw = w^3/3 - w + C = \cos^3(u)/3 - \cos u + C && (41)
\end{aligned}$$

if $w=\cos u$, $dw=-\sin u\,du$, so $-dw=\sin u\,du$. Note that since $x=\sin u$, we have $\sqrt{1-x^2}=\cos u$ (drawing a triangle with hypotenuse 1 and opposite side x).

Thus $\int \frac{x^3}{\sqrt{1-x^2}}\,dx = (\sqrt{1-x^2})^3/3 - \sqrt{1-x^2} + C$, so

$$\begin{aligned}
\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\,dx &= \lim_{a\to 1^-}[(\sqrt{1-x^2})^3/3 - \sqrt{1-x^2}]_0^a && (42)\\
&= \lim_{a\to 1^-}(\sqrt{1-a^2})^3/3 - \sqrt{1-a^2} - (\sqrt{1}^3/3 - \sqrt{1}) && (43)\\
&= 1 - \frac{1}{3} = \frac{2}{3}. && (44)
\end{aligned}$$